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ON THE EXISTENCE OF FELLER SEMIGROUPS WITH BOUNDARY CONDITIONS II

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Dedicated to Professor Hikosaburo Komatsu on the occasion of his 60th birthday

Abstract. This paper is devoted to the functional analytic approach to the problem of construction of Feller semigroups with Ventcel' (Wentzell) boundary conditions. The problem of construction of Feller semigroups has never before been studied in the *characteristic* case. In this paper we consider the characteristic case. Intuitively, our result may be stated as follows: We can construct a Feller semigroup corresponding to such a diffusion phenomenon that one of the reflection and viscosity phenomena occurs at each point of the boundary.

Introduction

Let D be a bounded domain of Euclidean space $\mathbf{R}^N$, with C^∞ boundary ∂D, and let $C(\overline{D})$ be the space of real-valued, continuous functions on the closure $\overline{D} = D \cup \partial D$. We equip the space $C(\overline{D})$ with the topology of uniform convergence on the whole $\overline{D}$; hence it is a Banach space with the maximum norm

$$\|f\| = \max_{x \in \overline{D}} |f(x)|.$$

A strongly continuous semigroup $\{T_t\}_{t \geq 0}$ on the space $C(\overline{D})$ is called a *Feller semigroup* on $\overline{D}$ if it is non-negative and contractive on $C(\overline{D})$:

$$f \in C(\overline{D}),\ 0 \leq f \leq 1 \quad \text{on } \overline{D} \implies 0 \leq T_t f \leq 1 \quad \text{on } \overline{D}.$$

Let A be a second-order, *degenerate* elliptic differential operator with real coefficients such that

$$Au(x) = \sum_{i,j=1}^{N} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{N} b^i(x) \frac{\partial u}{\partial x_i}(x) + c(x)u(x),$$

where:

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(1) $a^{ij} \in C^\infty(\mathbf{R}^N)$, $a^{ij} = a^{ji}$ and

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geq 0, \quad x \in \mathbf{R}^N, \ \xi \in \mathbf{R}^N.$$

(2) $b^i \in C^\infty(\mathbf{R}^N)$.
(3) $c \in C^\infty(\mathbf{R}^N)$ and $c \leq 0$ on $\overline{D}$.

The differential operator A is called a *diffusion operator* which describes analytically a strong Markov process with continuous paths (diffusion process) in the interior D.

This paper is devoted to the functional analytic approach to the problem of construction of Feller semigroups with Ventcel' boundary conditions, generalizing some results of Taira [Ta2] to the case when the operator A is *characteristic* with respect to the boundary ∂D as in Taira [Ta3], which we formulate precisely.

First we introduce a function b on the boundary ∂D by the formula:

$$b(x') = \sum_{i=1}^{N} \left(b^i(x') - \sum_{j=1}^{N} \frac{\partial a^{ij}}{\partial x_j}(x') \right) n_i, \quad x' \in \partial D,$$

where $\mathbf{n} = (n_1, \cdots, n_N)$ is the unit interior normal to ∂D at x'. The function b is called the *Fichera function* for the operator A. We divide the boundary ∂D into the following four disjoint subsets:

$$\Sigma_3 = \left\{ x' \in \partial D; \sum_{i,j=1}^{N} a^{ij}(x') n_i n_j > 0 \right\}.$$

$$\Sigma_2 = \left\{ x' \in \partial D; \sum_{i,j=1}^{N} a^{ij}(x') n_i n_j = 0, b(x') < 0 \right\}.$$

$$\Sigma_1 = \left\{ x' \in \partial D; \sum_{i,j=1}^{N} a^{ij}(x') n_i n_j = 0, b(x') > 0 \right\}.$$

$$\Sigma_0 = \left\{ x' \in \partial D; \sum_{i,j=1}^{N} a^{ij}(x') n_i n_j = 0, b(x') = 0 \right\}.$$

The fundamental hypothesis for A is the following (see Figure 1):

(H) Each set Σ_i $(i = 0, 1, 2, 3)$ consists of a finite number of connected *hypersurfaces*.

Intuitively, hypothesis (H) implies that a Markovian particle moves continuously in the interior D and may reach the boundary $\Sigma_2 \cup \Sigma_3$; so one may impose a boundary condition only on the set $\Sigma_2 \cup \Sigma_3$ (cf. [OR], [SV]).

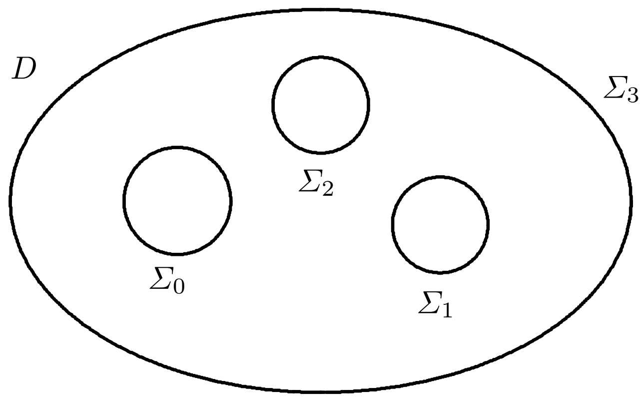

Figure 1

Furthermore we assume that, in a neighborhood of the boundary Σ_3, there exists a real-valued, C^∞ function Φ on $\mathbf{R}^N$ such that

$$\begin{aligned}
&D = \left\{x \in \mathbf{R}^N ; \Phi(x) > 0\right\}, \\
&\Sigma_3 = \left\{x \in \mathbf{R}^N ; \Phi(x) = 0\right\}, \\
&\operatorname{grad} \Phi(x) \neq 0 \quad \text{on } \Sigma_3.
\end{aligned}$$

Let Λ be a real C^∞ vector field on $\mathbf{R}^N$ such that

$$\Lambda \Phi = 1 \quad \text{in a neighborhood of } \Sigma_3.$$

Then we assume that, in a neighborhood of Σ_3, the operator A can be written in the following form:

$$Au = \Lambda^*(\Lambda u) + \Phi^{2k}(Pu) + \Phi^k \Lambda(Qu) + \Phi^{k-1}(Ru) + E(\Lambda u) + Fu,$$

where:

(1) k is a positive integer.

(2) P is a second-order differential operator acting along the surfaces parallel to the boundary Σ_3.

(3) Q and R are first-order differential operators acting along the surfaces parallel to the boundary Σ_3.

(4) E and F are C^∞ functions on $\mathbf{R}^N$.

Our fundamental hypothesis for the operator A near the boundary Σ_3 is stated as follows:

(E) The differential operator $\Lambda^*\Lambda + P + \Lambda Q$ is *elliptic* near the boundary Σ_3.

Let L be a second-order, boundary condition with real coefficients such that in local coordinates $(x_1, x_2, \cdots, x_{N-1})$ on $\Sigma_2 \cup \Sigma_3$

$$Lu(x') = \left(\sum_{i,j=1}^{N-1} \alpha^{ij}(x') \frac{\partial^2 u}{\partial x_i \partial x_j}(x') + \sum_{i=1}^{N-1} \beta^i(x') \frac{\partial u}{\partial x_i}(x')\right)$$

$$+\gamma(x')u(x') + \mu(x')\frac{\partial u}{\partial \mathbf{n}}(x') - \delta(x')Au(x'),$$

where:

(1) The α^{ij} are the components of a C^∞ symmetric contravariant tensor of type $\binom{2}{0}$ on boundary $\Sigma_2 \cup \Sigma_3$ and

$$\sum_{i,j=1}^{N-1} \alpha^{ij}(x')\xi_i\xi_j \geq 0, \quad x' \in \Sigma_2 \cup \Sigma_3, \quad \xi = \sum_{j=1}^{N-1} \xi_j \, dx_j \in T^*_{x'}(\Sigma_2 \cup \Sigma_3).$$

Here $T^*_{x'}(\Sigma_2 \cup \Sigma_3)$ is the cotangent space of $\Sigma_2 \cup \Sigma_3$ at x'.

(2) $\beta^i \in C^\infty(\Sigma_2 \cup \Sigma_3)$.

(3) $\gamma \in C^\infty(\Sigma_2 \cup \Sigma_3)$ and $\gamma \leq 0$ on $\Sigma_2 \cup \Sigma_3$.

(4) $\mu \in C^\infty(\Sigma_2 \cup \Sigma_3)$ and $\mu \geq 0$ on $\Sigma_2 \cup \Sigma_3$.

(5) $\delta \in C^\infty(\Sigma_2 \cup \Sigma_3)$ and $\delta \geq 0$ on $\Sigma_2 \cup \Sigma_3$.

(6) $\mathbf{n} = (n_1, \cdots, n_N)$ is the unit interior normal to the boundary $\Sigma_2 \cup \Sigma_3$.

The boundary condition L is called a second-order *Ventcel' boundary condition*. The terms of L are supposed to correspond to the diffusion along the boundary, the absorption phenomenon, the reflection phenomenon and the viscosity phenomenon, respectively.

We say that the boundary condition L is *transversal* on the boundary $\Sigma_2 \cup \Sigma_3$ if it satisfies the condition:

$$\mu + \delta > 0 \quad \text{on } \Sigma_2 \cup \Sigma_3.$$

Intuitively, the transversality condition implies that either reflection or viscosity phenomenon occurs on the boundary $\Sigma_2 \cup \Sigma_3$.

It is known (cf. [BCP], [Ta1]) that the infinitesimal generator $\mathfrak{A}$ of a Feller semigroup $\{T_t\}_{t\geq 0}$ is described analytically by a diffusion operator A and a Ventcel' boundary condition L.

In this paper, we consider the following problem:

Problem. *Conversely, given analytic data (A, L), can we construct a Feller semigroup $\{T_t\}_{t\geq 0}$ whose infinitesimal generator $\mathfrak{A}$ is characterized by (A, L) ?*

Our main result asserts that there exists a Feller semigroup on $\overline{D}$ corresponding to such a diffusion phenomenon that one of the reflection and viscosity phenomena occurs at each point of the boundary:

Main Theorem. *Assume that the operator A satisfies hypotheses (H) and (E) and that the boundary condition L is* transversal *on $\Sigma_2 \cup \Sigma_3$. Then there exists a Feller semigroup $\{T_t\}_{t\geq 0}$ on $\overline{D}$ whose infinitesimal generator $\mathfrak{A}$ is characterized as follows:*

(1) The domain $D(\mathfrak{A})$ of $\mathfrak{A}$ is the space

$$D(\mathfrak{A}) = \left\{u \in C(\overline{D}); Au \in C(\overline{D}), Lu = 0 \text{ on } \Sigma_2 \cup \Sigma_3\right\}.$$

(2) $\mathfrak{A}u = Au$, $u \in D(\mathfrak{A})$.

Here Au and Lu are taken in the sense of distributions.

We remark that Cattiaux [Ca] has proved a probabilistic version of Main Theorem in the non-characteristic case: $\partial D = \Sigma_3$.

The rest of this paper is organized as follows.

Sections 1 and 2 provide a review of basic results about Feller semigroups and pseudo-differential operators which will be used in the subsequent sections.

In Section 3 we consider the following Dirichlet problem:

$$(D) \qquad \begin{cases} (\alpha - A)u = f & \text{in } D, \\ u = \varphi & \text{on } \Sigma_2 \cup \Sigma_3, \end{cases}$$

where α ia a positive parameter. The existence and uniqueness theorem for problem (D) is well established in the framework of Hölder spaces (Theorem 3.2).

The purpose of Section 4 is to give a general existence theorem for Feller semigroups in terms of boundary value problems (Theorem 4.9). In other words, we reduce the problem of construction of Feller semigroups to the problem of unique solvability for the boundary value problem

$$(*) \qquad \begin{cases} (\alpha - A)u = f & \text{in } D, \\ Lu = 0 & \text{on } \Sigma_2 \cup \Sigma_3. \end{cases}$$

In Section 5 we prove an existence theorem for problem $(*)$ in the framework of Hölder spaces (Theorem 5.1). The idea of our approach is stated as follows.

First we consider the Dirichlet problem:

$$\begin{cases} (\alpha - A)v = f & \text{in } D, \\ v = 0 & \text{on } \Sigma_2 \cup \Sigma_3, \end{cases}$$

and let

$$v = G_\alpha^0 f.$$

Then it follows that a function u is a solution of the problem $(*)$ if and only if the function $w = u - v = u - G_\alpha^0 f$ is a solution of the problem

$$\begin{cases} (\alpha - A)w = 0 & \text{in } D, \\ Lw = -Lv = -LG_\alpha^0 f & \text{on } \Sigma_2 \cup \Sigma_3. \end{cases}$$

But every solution w of the homogeneous equation $(\alpha - A)w = 0$ in D can be expressed in the form:

$$w = H_\alpha \psi.$$

Thus, by using the operators G_α^0 and H_α, one can reduce the study of problem $(*)$ to that of the equation:

$$LH_\alpha \psi = -LG_\alpha^0 f \quad \text{on } \Sigma_2 \cup \Sigma_3.$$

If hypothesis (E) is satisfied, then it follows that the operator LH_α is a second-order, *pseudo-differential operator* on the boundary $\Sigma_2 \cup \Sigma_3$. More precisely, the operator LH_α is the sum of a second-order, degenerate elliptic differential operator Q_α and a pseudo-differential operator Π_α, where Π_α is a elliptic pseudo-differential operator of order $1/(k+1)$ on Σ_3, and is a second-order, degenerate elliptic differential operator on Σ_2. This is the essential step in the proof of Main Theorem.

By using the theory of pseudo-differential operators, we can show that if the boundary condition L is transversal on $\Sigma_2 \cup \Sigma_3$, then the operator LH_α is *bijective* in the framework of Hölder spaces.

Summing up, we find that a unique solution u of problem $(*)$ can be expressed as follows:

$$u = G_\alpha^0 f - H_\alpha \left((LH_\alpha)^{-1} \left(LG_\alpha^0 f \right) \right). \tag{$**$}$$

Section 6 is devoted to the proof of Main Theorem (Theorem 6.1). By virtue of formula $(**)$, we can verify all the conditions of the generation theorem of Feller semigroups, just as in the proof of [Ta2, Theorem 3.16].

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1. Theory of Feller Semigroups

This section provides a review of basic results about Feller semigroups, which forms a functional analytic background for the proof of Main Theorem.

1.1 Feller Semigroups. Let K be a compact metric space and let $C(K)$ be the space of real-valued, continuous functions on K. The space $C(K)$ is a Banach space with the maximum norm

$$\|f\| = \max_{x \in K} |f(x)|.$$

A family $\{T_t\}_{t \geq 0}$ of bounded linear operators acting on $C(K)$ is called a *Feller semigroup* on K if it satisfies the following three conditions:

(i) $T_{t+s} = T_t \cdot T_s$, t, $s \geq 0$; $T_0 = I$.

(ii) The family $\{T_t\}$ is strongly continuous in t for $t \geq 0$:

$$\lim_{s \downarrow 0} \|T_{t+s} f - T_t f\| = 0, \quad f \in C(K).$$

(iii) The family $\{T_t\}$ is non-negative and contractive on $C(K)$:

$$f \in C(K),\, 0 \leq f \leq 1 \quad \text{on } K \implies 0 \leq T_t f \leq 1 \quad \text{on } K.$$

1.2 Generation Theorems of Feller Semigroups. If $\{T_t\}_{t \geq 0}$ is a Feller semigroup on K, we define its *infinitesimal generator* $\mathfrak{A}$ by the formula

$$\mathfrak{A} u = \lim_{t \downarrow 0} \frac{T_t u - u}{t}, \tag{1.1}$$

provided that the limit (1.1) exists in $C(K)$.

The next theorem is a version of the Hille-Yosida theorem adapted to the present context (cf. [Ta1, Theorem 9.3.1]):

Theorem 1.1. *(i) Let $\{T_t\}_{t\geq 0}$ be a Feller semigroup on K and $\mathfrak{A}$ its infinitesimal generator. Then we have the following assertions:*

(a) The domain $D(\mathfrak{A})$ is everywhere dense in the space $C(K)$.

(b) For each $\alpha > 0$, the equation $(\alpha I - \mathfrak{A})u = f$ has a unique solution u in $D(\mathfrak{A})$ for any $f \in C(K)$. Hence, for each $\alpha > 0$, the Green operator $(\alpha I - \mathfrak{A})^{-1} : C(K) \to C(K)$ can be defined by the formula

$$u = (\alpha I - \mathfrak{A})^{-1} f, \quad f \in C(K).$$

(c) For each $\alpha > 0$, the operator $(\alpha I - \mathfrak{A})^{-1}$ is non-negative on the space $C(K)$:

$$f \in C(K)\,,\, f \geq 0 \quad \text{on } K \implies (\alpha I - \mathfrak{A})^{-1} f \geq 0 \quad \text{on } K.$$

(d) For each $\alpha > 0$, the operator $(\alpha I - \mathfrak{A})^{-1}$ is bounded on the space $C(K)$ with norm

$$\|(\alpha I - \mathfrak{A})^{-1}\| \leq \frac{1}{\alpha}\,.$$

(ii) Conversely, if $\mathfrak{A}$ is a linear operator from $C(K)$ into itself satisfying condition (a) and if there is a constant $\alpha_0 \geq 0$ such that, for all $\alpha > \alpha_0$, conditions (b) through (d) are satisfied, then $\mathfrak{A}$ is the infinitesimal generator of some Feller semigroup $\{T_t\}_{t\geq 0}$ on K.

We give useful criteria in terms of the *maximum principle* (cf. [BCP, Théorème de Hille-Yosida-Ray], [Ta1, Theorem 9.3.3]):

Theorem 1.2. *(i) Let B be a linear operator from the space $C(K)$ into itself, and assume that:*

(α) The domain $D(B)$ of B is everywhere dense in the space $C(K)$.

(β) There exists an open and dense subset K_0 of K such that if $u \in D(B)$ takes a positive *maximum at a point x_0 of K_0, then we have*

$$Bu(x_0) \leq 0.$$

Then the operator B is closable in the space $C(K)$.

(ii) Let B be as in part (i), and further assume that:

(β') If $u \in D(B)$ takes a positive *maximum at a point x_0' of K, then we have*

$$Bu(x_0') \leq 0.$$

(γ) For some $\alpha_0 \geq 0$, the range $R(\alpha_0 I - B)$ of $\alpha_0 I - B$ is everywhere dense in the space $C(K)$.

Then the minimal closed extension $\overline{B}$ of B is the infinitesimal generator of some Feller semigroup on K.

2. Pseudo-Differential Operators

This section provides a review of basic results about pseudo-differential operators which will be used in the subsequent sections.

2.1 Function Spaces. If Ω is an open subset of $\mathbf{R}^n$, then we let

$$L^\infty(\Omega) = \text{the space of equivalence classes of essentially bounded, Lebesgue measurable functions on } \Omega.$$

If m is a non-negative integer, we let

$$\begin{aligned} C^m(\Omega) &= \text{the space of functions of class } C^m \text{ on } \Omega, \\ C_0^m(\Omega) &= \text{the space of functions in } C^m(\Omega) \text{ with compact support in } \Omega, \end{aligned}$$

and

$$C^m(\overline{\Omega}) = \text{the space of functions in } C^m(\Omega) \text{ all of whose derivatives of order } \leq m \text{ have continuous extensions to the closure } \overline{\Omega}.$$

If Ω is bounded, then $C^m(\overline{\Omega})$ is a Banach space with the norm

$$\|u\|_{C^m(\overline{\Omega})} = \max_{\substack{x \in \overline{\Omega} \\ |\alpha| \leq m}} |\partial^\alpha u(x)|.$$

If m is a non-negative integer and $0 < \theta < 1$, we let

$$C^{m+\theta}(\Omega) = \text{the space of functions in } C^m(\Omega) \text{ all of whose } m\text{-th order derivatives are locally Hölder continuous with exponent } \theta \text{ on } \Omega,$$

and

$$C^{m+\theta}(\overline{\Omega}) = \text{the space of functions in } C^m(\overline{\Omega}) \text{ all of whose } m\text{-th order derivatives are Hölder continuous with exponent } \theta \text{ on } \overline{\Omega}.$$

If Ω is bounded, then $C^{m+\theta}(\overline{\Omega})$ is a Banach space with the norm

$$\|u\|_{C^{m+\theta}(\overline{\Omega})} = \|u\|_{C^m(\overline{\Omega})} + \max_{|\alpha|=m} \sup_{\substack{x,y \in \overline{\Omega} \\ x \neq y}} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^\theta}.$$

If M is an n-dimensional, compact C^∞ manifold without boundary, then the space $C^{m+\theta}(M)$ is defined to be locally the space $C^{m+\theta}(\mathbf{R}^n)$, upon using local coordinate systems flattening out M, together with a partition of unity. The norm of the space $C^{m+\theta}(M)$ will be denoted by $\|\cdot\|_{C^{m+\theta}(M)}$.

2.2 Unique Solvability Theorem for Pseudo-Differential Operators. The next result will play an essential role in the proof of an existence theorem for degenerate elliptic boundary value problems in the framework of Hölder spaces in Section 5 (cf. [Ta2, Theorem 2.1]):

Theorem 2.1. *Let T be a second-order, classical pseudo-differential operator on an n-dimensional, compact C^∞ manifold M without boundary such that*

$$T = P + S,$$

where:

(a) The operator P is a second-order, degenerate elliptic differential operator *on M with non-positive principal symbol, and $P1 \leq 0$ on M.*

(b) The operator S is a classical pseudo-differential operator of order $2-\kappa$, $\kappa > 0$, on M and its distribution kernel $s(x,y)$ is non-negative *off the diagonal in $M \times M$.*

(c) $T1 = P1 + S1 \leq 0$ on M.

Then, for each integer $k \geq 1$, there exists a constant $\lambda = \lambda(k) > 0$ such that, for any $f \in C^{k+\theta}(M)$ with $0 < \theta < 1$, one can find a function $\varphi \in C^{k+\theta}(M)$ satisfying

$$(T - \lambda)\varphi = f \quad \text{on } M,$$

and

$$\|\varphi\|_{C^{k+\theta}(M)} \leq C_{k+\theta} \|f\|_{C^{k+\theta}(M)} .$$

Here $C_{k+\theta} > 0$ is a constant independent of f.

2.3 Positive Borel Kernels. If Ω is an open subset of $\mathbf{R}^n$, we let

$$\begin{aligned} B_{loc}(\Omega) = &\text{the space of Borel-measurable functions in } \Omega \\ &\text{which are bounded on compact subsets of } \Omega. \end{aligned}$$

Let $\mathcal{B}$ be the σ-algebra of all Borel sets in Ω. A *positive Borel kernel* on Ω is a mapping

$$x \longmapsto s(x, dy)$$

of Ω into the space of non-negative measures on $\mathcal{B}$ such that, for each $X \in \mathcal{B}$, the function $s(\cdot, X)$ is Borel-measurable on Ω.

Now we assume that a positive Borel kernel $s(x, dy)$ satisfies the following two conditions:

(NS1) $s(x, \{x\}) = 0$ for all $x \in \Omega$.

(NS2) For all non-negative functions f in $C(\Omega)$ with compact support, the function

$$x \longmapsto \int_\Omega s(x, dy)|y - x|^2 f(y), \quad x \in \Omega,$$

belongs to the space $B_{loc}(\Omega)$.

Let $\sigma(x,y)$ be a C^∞ function on $\Omega \times \Omega$ such that:

(a) $0 \leq \sigma(x,y) \leq 1$ on $\Omega \times \Omega$.

(b) $\sigma(x,y) = 1$ in a neighborhood of the diagonal in $\Omega \times \Omega$.

(c) For any compact subset K of Ω, there exists a compact subset K' of Ω such that the functions $\sigma_x(\cdot) = \sigma(x, \cdot)$, $x \in K$, have their support in K'.

Then we can define a linear operator

$$S : C_0^2(\Omega) \longrightarrow B_{loc}(\Omega)$$

by the formula

$$Su(x) = \int_\Omega s(x, dy) \left[u(y) - \sigma(x,y) \left(u(x) + \sum_{i=1}^n \frac{\partial u}{\partial x_i}(x)(y_i - x_i) \right) \right].$$

The next theorem gives a useful characterization of positive Borel kernels (cf. [BCP, Théorème I]):

Theorem 2.2. *Let A be a linear operator from $C_0^2(\Omega)$ into $B_{loc}(\Omega)$. Then the following two assertions are equivalent:*

(i) $A : C_0^2(\Omega) \to B_{loc}(\Omega)$ is continuous and satisfies the condition

$$x \in \Omega, u \in C_0^2(\Omega), u \geq 0 \text{ in } \Omega \text{ and } x \notin \operatorname{supp} u \implies Au(x) \geq 0.$$

(ii) There exist a second-order differential operator $P : C^2(\Omega) \to B_{loc}(\Omega)$ and a positive Borel kernel $s(x, dy)$, having properties (NS1) and (NS2), such that the operator A is written of the form

$$Au(x) = Pu(x) + Su(x), \quad x \in \Omega, u \in C_0^2(\Omega).$$

3. The Dirichlet Problem

In this section we shall study the Dirichlet problem for second-order, degenerate elliptic differential operators in the framework of Hölder spaces.

We consider the following Dirichlet problem: For given functions $f \in L^\infty(D)$ and $g \in L^\infty(\Sigma_2 \cup \Sigma_3)$, find a function $u \in L^\infty(D)$ such that

$$\begin{cases} Au = f & \text{in } D, \\ u = g & \text{on } \Sigma_2 \cup \Sigma_3. \end{cases} \tag{+}$$

Now we give the precise definition of a weak solution of problem (+):

Definition 3.1. A function $u \in L^\infty(D)$ is called a *weak solution* of problem (+) if we have, for any function $v \in C^2(\overline{D})$ satisfying $v = 0$ on $\Sigma_1 \cup \Sigma_3$,

$$\iint_D u \cdot A^* v \, dx = \iint_D f v \, dx - \int_{\Sigma_3} g \frac{\partial v}{\partial \nu} \, d\sigma + \int_{\Sigma_2} b g v \, d\sigma.$$

Here A^* is the formal adjoint operator for A, $\partial/\partial\nu = \sum_{i,j} a^{ij} n_j \partial/\partial x_i$ is the conormal derivative associated with A, b is the Fichera function for A, and $d\sigma$ is the surface element of $\Sigma_2 \cup \Sigma_3$.

The next theorem states the existence and uniqueness theorem for the Dirichlet problem in the framework of Hölder spaces (cf. [OR, Theorem 1.8.2], [Ta3, Theorem 2]):

Theorem 3.2. *Assume that hypothesis (H) is satisfied and that*

$$c < 0 \quad \text{on } \overline{D}$$

and

$$c^* = \sum_{i,j=1}^N \frac{\partial^2 a^{ij}}{\partial x_i \partial x_j} - \sum_{i=1}^N \frac{\partial b^i}{\partial x_i} + c < 0 \quad \text{on } \overline{D}.$$

Then, for each integer $m \geq 2$, one can find a constant $\lambda(m) > 0$ such that, for each $\lambda \geq \lambda(m)$, the Dirichlet problem

$$\begin{cases} (A - \lambda)u = f & \text{in } D, \\ u = \varphi & \text{on } \Sigma_2 \cup \Sigma_3 \end{cases}$$

has a unique solution u in $C^{m+\theta}(\overline{D})$ for any $f \in C^{2m+2+2\theta}(\overline{D})$ and any $\varphi \in C^{2m+4+2\theta}(\Sigma_2 \cup \Sigma_3)$ with $0 < \theta < 1$.

4. Feller Semigroups and Boundary Value Problems

The purpose of this section is to give a general existence theorem for Feller semigroups in terms of boundary value problems (Theorem 4.9), generalizing some results in [Ta1, Section 9.6].

4.1 Green and Harmonic Operators. First we consider the following Dirichlet problem:

$$\text{(D)} \qquad \begin{cases} (\alpha - A)u = f & \text{in } D, \\ u = \varphi & \text{on } \Sigma_2 \cup \Sigma_3, \end{cases}$$

where $\alpha > 0$ is a parameter.

If we take a parameter α so large that

$$c^* - \alpha = \sum_{i,j=1}^{N} \frac{\partial^2 a^{ij}}{\partial x_i \partial x_j} - \sum_{i=1}^{N} \frac{\partial b^i}{\partial x_i} + c - \alpha < 0 \quad \text{on } \overline{D}$$

and also

$$\alpha \geq \lambda(m),$$

where $\lambda(m)$ is the constant stated in Theorem 3.2, then we have the following existence and uniqueness theorem for problem (D) in the framework of Hölder spaces:

Theorem 4.1. *Assume that hypothesis (H) is satisfied. Then, for each integer $m \geq 2$, one can find a constant $\alpha(m) > 0$ such that, for each $\alpha \geq \alpha(m)$, problem (D) has a unique solution u in $C^{m+\theta}(\overline{D})$ for any $f \in C^{2m+2+2\theta}(\overline{D})$ and any $\varphi \in C^{2m+4+2\theta}(\Sigma_2 \cup \Sigma_3)$ with $0 < \theta < 1$.*

Now, by using Theorem 4.1 with $m = 2$ and $\alpha \geq \alpha(2)$, we can introduce linear operators

$$G_\alpha^0 : C^{6+2\theta}(\overline{D}) \longrightarrow C^{2+\theta}(\overline{D})$$

and

$$H_\alpha : C^{8+2\theta}(\Sigma_2 \cup \Sigma_3) \longrightarrow C^{2+\theta}(\overline{D})$$

as follows.

(a) For any $f \in C^{6+2\theta}(\overline{D})$, the function $G_\alpha^0 f \in C^{2+\theta}(\overline{D})$ is the unique solution of the problem:

$$(4.1) \qquad \begin{cases} (\alpha - A)G_\alpha^0 f = f & \text{in } D, \\ G_\alpha^0 f = 0 & \text{on } \Sigma_2 \cup \Sigma_3. \end{cases}$$

(b) For any $\varphi \in C^{8+2\theta}(\Sigma_2 \cup \Sigma_3)$, the function $H_\alpha \varphi \in C^{2+\theta}(\overline{D})$ is the unique solution of the problem:

$$(4.2) \qquad \begin{cases} (\alpha - A)H_\alpha \varphi = 0 & \text{in } D, \\ H_\alpha \varphi = \varphi & \text{on } \Sigma_2 \cup \Sigma_3. \end{cases}$$

The operator G_α^0 is called the *Green operator* and the operator H_α is called the *harmonic operator*, respectively.

Then we have the following results (cf. [Ta1, Theorem 9.6.4]):

Theorem 4.2. *(i) The operator G_α^0 can be uniquely extended to a non-negative, bounded linear operator on $C(\overline{D})$ into itself, denoted again G_α^0, with norm $\|G_\alpha^0\| \leq 1/\alpha$.*

(ii) The operator H_α can be uniquely extended to a non-negative, bounded linear operator on $C(\Sigma_2 \cup \Sigma_3)$ into $C(\overline{D})$, denoted again H_α, with norm $\|H_\alpha\| = 1$.

4.2 Existence Theorem for Feller Semigroups. Now we consider the following boundary value problem (∗) in the framework of the spaces of *continuous functions*:

$$(*) \qquad \begin{cases} (\alpha - A)u = f & \text{in } D, \\ Lu = \varphi & \text{on } \Sigma_2 \cup \Sigma_3. \end{cases}$$

To do this, we introduce three operators associated with problem (∗).

(I) First we introduce a linear operator

$$A : C(\overline{D}) \longrightarrow C(\overline{D})$$

as follows.

(a) The domain $D(A)$ of A is the space $C^2(\overline{D})$.

(b) $Au = \sum_{i,j} a^{ij} \partial^2 u/\partial x_i \partial x_j + \sum_i b^i \partial u/\partial x_i + cu$, $u \in D(A)$.

Then we have the following (cf. [Ta1, Lemma 9.6.5]):

Lemma 4.3. *The operator A has its minimal closed extension $\overline{A}$ in the space $C(\overline{D})$.*

The extended operators $G_\alpha^0 : C(\overline{D}) \to C(\overline{D})$ and $H_\alpha : C(\Sigma_2 \cup \Sigma_3) \to C(\overline{D})$ still satisfy formulas (4.1) and (4.2) respectively in the following sense (cf. [Ta1, Lemma 9.6.7 and Corollary 9.6.8]):

Lemma 4.4. *(i) For any $f \in C(\overline{D})$, we have*

$$\begin{cases} G_\alpha^0 f \in D\left(\overline{A}\right), \\ \left(\alpha I - \overline{A}\right) G_\alpha^0 f = f \quad \textit{in } D. \end{cases}$$

(ii) For any $\varphi \in C(\Sigma_2 \cup \Sigma_3)$, we have

$$\begin{cases} H_\alpha \varphi \in D\left(\overline{A}\right), \\ \left(\alpha I - \overline{A}\right) H_\alpha \varphi = 0 \quad \textit{in } D. \end{cases}$$

Here $D\left(\overline{A}\right)$ is the domain of the closed extension $\overline{A}$.

Corollary 4.5. *Every function u in $D\left(\overline{A}\right)$ can be written in the following form:*

$$u = G_\alpha^0 \left(\left(\alpha I - \overline{A}\right) u\right) + H_\alpha \left(u|_{\Sigma_2 \cup \Sigma_3}\right).$$

(II) Secondly we introduce a linear operator

$$LG_\alpha^0 : C(\overline{D}) \longrightarrow C(\Sigma_2 \cup \Sigma_3)$$

as follows.

(a) The domain $D\left(LG_\alpha^0\right)$ of LG_α^0 is the space $C^{6+2\theta}(\overline{D})$.

(b) $LG_\alpha^0 f = L\left(G_\alpha^0 f\right),\ f \in D\left(LG_\alpha^0\right)$.

Then we have the following (cf. [Ta1, Lemma 9.6.9]):

Lemma 4.6. *The operator LG_α^0 can be uniquely extended to a non-negative, bounded linear operator $\overline{LG_\alpha^0} : C(\overline{D}) \to C\left(\Sigma_2 \cup \Sigma_3\right)$.*

(III) Finally we introduce a linear operator

$$LH_\alpha : C\left(\Sigma_2 \cup \Sigma_3\right) \longrightarrow C\left(\Sigma_2 \cup \Sigma_3\right)$$

as follows.

(a) The domain $D\left(LH_\alpha\right)$ of LH_α is the space $C^{8+2\theta}\left(\Sigma_2 \cup \Sigma_3\right)$.
(b) $LH_\alpha \psi = L\left(H_\alpha \psi\right), \ \psi \in D\left(LH_\alpha\right)$.

Then we have the following:

Lemma 4.7. *The operator LH_α has its minimal closed extension $\overline{LH_\alpha}$ in the space $C\left(\Sigma_2 \cup \Sigma_3\right)$.*

Proof. We apply part (i) of Theorem 1.2 to the operator LH_α. To do this, it suffices to show that the operator LH_α satisfies condition (β) of the same theorem.

Assume that a function ψ in the domain $D\left(LH_\alpha\right) = C^{8+2\theta}\left(\Sigma_2 \cup \Sigma_3\right)$ takes its *positive* maximum at some point x_0' of $\Sigma_2 \cup \Sigma_3$. Then it follows that the function $H_\alpha\psi$ is in $C^{2+\theta}(\overline{D})$ and satisfies:

$$\begin{cases} (A-\alpha)H_\alpha\psi = 0 & \text{in } D, \\ H_\alpha\psi = \psi & \text{on } \Sigma_2 \cup \Sigma_3. \end{cases}$$

Hence, applying the weak maximum principle (cf. [Ta2, Theorem A.1]) to our situation, we find that the function $H_\alpha\psi$ takes its positive maximum $\psi(x_0')$ at $x_0' \in \Sigma_2 \cup \Sigma_3$.

The next claim is the essential step in the proof of Lemma 4.7:

Claim 4.8. *The interior normal derivative $(\partial/\partial\mathbf{n})\left(H_\alpha\psi\right)$ of the function $H_\alpha\psi$ satisfies the condition*

$$\frac{\partial}{\partial\mathbf{n}}\left(H_\alpha\psi\right) < 0 \quad \text{on } \Sigma_2 \cup \Sigma_3.$$

Proof. (i) First let x_0' be a point of Σ_3. Since the function $H_\alpha\,\psi$ takes its positive maximum $\psi(x_0')$ at $x_0' \in \Sigma_3$, we can apply the boundary point lemma (cf. [Ta2, Lemma A.3]) to obtain that

$$\frac{\partial}{\partial\mathbf{n}}\left(H_\alpha\psi\right)\left(x_0'\right) < 0.$$

(ii) Next, if x_0' is a point of Σ_2, then we choose a local coordinate system $(y_1, \cdots, y_{N-1}, y_N)$ in a neighborhood of x_0' such that

$$\begin{cases} x_0' = 0, \\ D = \left\{y_N > 0\right\}, \\ \Sigma_2 = \left\{y_N = 0\right\}, \end{cases}$$

and assume that, in terms of this coordinate system, the operator A is written in the form

$$A = a^{NN}(y)\frac{\partial^2}{\partial y_N^2} + b^N(y)\frac{\partial}{\partial y_N} + \sum_{i,j=1}^{N-1} a^{ij}(y)\frac{\partial^2}{\partial y_i \partial y_j} + \sum_{i=1}^{N-1} b^i(y)\frac{\partial}{\partial y_i} + c(y).$$

We remark that:

(1) $a^{NN}(0) = 0$ and $b^N(0) = b(0) < 0$, since $x_0' = 0 \in \Sigma_2$.
(2) $a^{ij} \in C^\infty\left(\mathbf{R}^N\right)$, $a^{ij} = a^{ji}$ and

$$\sum_{i,j=1}^{N} a^{ij}(y)\xi_i\xi_j \geq 0\,, \quad y \in \mathbf{R}^N,\ \xi \in \mathbf{R}^N.$$

Since we have

$$\begin{cases} (A-\alpha)H_\alpha\psi = 0 & \text{in } D, \\ H_\alpha\psi = \psi & \text{on } \Sigma_2 \cup \Sigma_3, \end{cases}$$

and since the function $H_\alpha\psi$ takes its positive maximum $\psi(0)$ at $x_0' = 0$, it follows that

$$\begin{aligned} 0 &= a^{NN}(0)\frac{\partial^2}{\partial y_N^2}\left(H_\alpha\psi\right)(0) + b^N(0)\frac{\partial}{\partial y_N}\left(H_\alpha\psi\right)(0) \\ &\quad + \sum_{i,j=1}^{N-1} a^{ij}(0)\frac{\partial^2\psi}{\partial y_i\partial y_j}(0) + \sum_{i=1}^{N-1} b^i(0)\frac{\partial\psi}{\partial y_i}(0) + \left(c(0)-\alpha\right)\psi(0) \\ &= b(0)\frac{\partial}{\partial y_N}\left(H_\alpha\psi\right)(0) + \sum_{i,j=1}^{N-1} a^{ij}(0)\frac{\partial^2\psi}{\partial y_i\partial y_j}(0) + \left(c(0)-\alpha\right)\psi(0). \end{aligned}$$

This proves that

$$\begin{aligned} \frac{\partial}{\partial \mathbf{n}}\left(H_\alpha\psi\right)(x_0') &= \frac{\partial}{\partial y_N}\left(H_\alpha\psi\right)(0) \\ &= -\frac{1}{b(0)}\left(\sum_{i,j=1}^{N-1} a^{ij}(0)\frac{\partial^2\psi}{\partial y_i\partial y_j}(0) + \left(c(0)-\alpha\right)\psi(0)\right) \\ &\leq -\frac{1}{b(x_0')}\left(c(x_0')-\alpha\right)\psi(x_0') \\ &< 0. \end{aligned}$$

The proof of Claim 4.8 is complete. $\triangledown$

Hence we have

$$\begin{aligned} LH_\alpha\psi(x_0') &= \sum_{i,j=1}^{N-1} \alpha^{ij}(x_0')\frac{\partial^2\psi}{\partial x_i\partial x_j}(x_0') + \mu(x_0')\frac{\partial}{\partial \mathbf{n}}\left(H_\alpha\psi\right)(x_0') \\ &\quad + \gamma(x_0')\psi(x_0') - \alpha\delta(x_0')\psi(x_0') \\ &\leq 0, \quad x_0' \in \Sigma_2 \cup \Sigma_3. \end{aligned}$$

This verifies condition (β) of Theorem 1.2. $\square$

Now we can give a general existence theorem for Feller semigroups on $\Sigma_2 \cup \Sigma_3$ in terms of boundary value problem $(*)$ (cf. [Ta1, Theorem 9.6.15]):

Theorem 4.9. *(i) If the operator $\overline{LH_\alpha}$ is the infinitesimal generator of a Feller semigroup on $\Sigma_2 \cup \Sigma_3$, then, for each constant $\lambda > 0$, the boundary value problem*

$$(*)_0 \qquad \begin{cases} (\alpha - A)u = 0 & \text{in } D, \\ (\lambda - L)u = \varphi & \text{on } \Sigma_2 \cup \Sigma_3 \end{cases}$$

has a solution $u \in C^{2+\theta}(\overline{D})$ for any φ in some dense subset of $C(\Sigma_2 \cup \Sigma_3)$.

(ii) Conversely, if, for some constant $\lambda \geq 0$, problem $(*)_0$ has a solution $u \in C^{8+2\theta}(\overline{D})$ for any φ in some dense subset of $C(\Sigma_2 \cup \Sigma_3)$, then the operator $\overline{LH_\alpha}$ is the infinitesimal generator of some Feller semigroup on $\Sigma_2 \cup \Sigma_3$.

5. Degenerate Elliptic Boundary Value Problems

Now we can prove an existence theorem for degenerate elliptic boundary value problems in the framework of Hölder spaces, which will play an important role in the proof of Main Theorem:

Theorem 5.1. *Assume that hypotheses (H) and (E) are satisfied. Then, for each $\alpha \geq \alpha(20)$, there exists a constant $\lambda = \lambda(\alpha) > 0$ such that the boundary value problem*

$$(*) \qquad \begin{cases} (\alpha - A)u = f & \text{in } D, \\ (\lambda - L)u = \varphi & \text{on } \Sigma_2 \cup \Sigma_3 \end{cases}$$

has a solution u in the space $C^{2+\theta}(\overline{D})$ for any $f \in C^{20+2\theta}(\overline{D})$ and any $\varphi \in C^{8+2\theta}(\Sigma_2 \cup \Sigma_3)$ with $0 < \theta < 1$. Here $\alpha(20)$ is the constant stated in Theorem 4.1 with $m = 20$.

Proof. We divide the proof into three steps.

(I) First we reduce the study of problem $(*)$ to that of an operator on the boundary $\Sigma_2 \cup \Sigma_3$.

Assume that:

(5.1) *If $\lambda > 0$ is sufficiently large, then the operator*

$$\lambda I - LH_\alpha : C^{8+2\theta}(\Sigma_2 \cup \Sigma_3) \longmapsto C^{8+2\theta}(\Sigma_2 \cup \Sigma_3)$$

is surjective.

Now let f be an arbitrary function in $C^{20+2\theta}(\overline{D})$ and φ an arbitrary function in $C^{8+2\theta}(\Sigma_2 \cup \Sigma_3)$. Then we have

$$G_\alpha^0 f \in C^{10+\theta}(\overline{D}),$$

and

$$LG_\alpha^0 f = \mu \frac{\partial}{\partial \mathbf{n}} \left(G_\alpha^0 f \right) + \delta f \in C^{9+\theta}(\Sigma_2 \cup \Sigma_3) \subset C^{8+2\theta}(\Sigma_2 \cup \Sigma_3).$$

Hence, by condition (5.1), one can find a function $\psi \in C^{8+2\theta}(\Sigma_2 \cup \Sigma_3)$ such that

$$(\lambda - LH_\alpha)\psi = \varphi + LG_\alpha^0 f \quad \text{on } \Sigma_2 \cup \Sigma_3.$$

If we let

$$u = G_\alpha^0 f + H_\alpha \psi \in C^{2+\theta}(\overline{D}),$$

then it follows that

$$(\alpha - A)u = f \quad \text{in } D,$$

and

$$(\lambda - L)u = -LG_\alpha^0 f + (\lambda - LH_\alpha)\,\psi = \varphi \quad \text{on } \Sigma_2 \cup \Sigma_3.$$

This proves that the function $u = G_\alpha^0 f + H_\alpha \psi$ is a solution of problem $(*)$.

Therefore, we are reduced to the study of the operator $\lambda I - LH_\alpha$ on the boundary $\Sigma_2 \cup \Sigma_3$.

(II) Next we show that the operator

$$\begin{aligned} LH_\alpha : C^{8+2\theta}(\Sigma_2 \cup \Sigma_3) &\longrightarrow C^\theta(\Sigma_2 \cup \Sigma_3) \\ \varphi &\longmapsto L\,(H_\alpha \varphi) \end{aligned}$$

is a second-order, classical *pseudo-differential operator* on the boundary $\Sigma_2 \cup \Sigma_3$, and satisfies all the conditions of Theorem 2.1.

We let

$$\begin{aligned} LH_\alpha\varphi = &\left[\sum_{i,j=1}^{N-1} \alpha^{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + \sum_{i=1}^{N-1} \beta^i \frac{\partial \varphi}{\partial x_i} + (\gamma - \alpha\delta)\varphi\right] \\ &+ \mu \frac{\partial}{\partial \mathbf{n}}(H_\alpha \varphi)\Big|_{\Sigma_2 \cup \Sigma_3} \\ := &\; Q_\alpha \varphi + \mu\, \Pi_\alpha \varphi. \end{aligned}$$

But we have the following results:

(1) The operator Q_α is a second-order, degenerate elliptic differential operator on $\Sigma_2 \cup \Sigma_3$. Note that

$$\sum_{i,j=1}^{N-1} \alpha^{ij}(x')\xi_i\xi_j \geq 0 \quad \text{on the cotangent bundle } T^*(\Sigma_2 \cup \Sigma_3),$$

and

$$Q_\alpha 1 = \gamma - \alpha\delta \leq 0 \quad \text{on } \Sigma_2 \cup \Sigma_3.$$

(2-a) If hypothesis (E) is satisfied, then it follows (cf. [De], [So]) that the operator Π_α is a classical, *elliptic pseudo-differential operator* of order $1/(k+1)$ on the boundary Σ_3, and its symbol is given by the following:

$$\left[p_{1/(k+1)}(x', \xi') + \sqrt{-1}\, q_{1/(k+1)}(x', \xi')\right] + \text{terms of order} \leq 0 \text{ depending on } \alpha,$$

where

$p_{1/(k+1)}(x', \xi') < 0$ on the bundle $T^*(\Sigma_3) \setminus \{0\}$ of non-zero cotangent vectors.

For example, if the operator A is of the form

$$A = \Lambda^* \Lambda + \Phi^{2k} \Delta',$$

where Δ' is the Laplace-Beltrami operator on Σ_3, then we have

$$p_{1/(k+1)}(x', \xi') = -(k+1) \left(\frac{2}{k+1} \right)^{\frac{1}{k+1}} \frac{\Gamma\left(\frac{k}{k+1}\right) \Gamma\left(\frac{3}{2(k+1)}\right)}{\Gamma\left(\frac{1}{k+1}\right) \Gamma\left(\frac{1}{2(k+1)}\right)} |\xi'|^{\frac{1}{k+1}}.$$

Here $|\xi'|$ is the length of ξ' with respect to the Riemannian metric of the boundary Σ_3 induced by the natural metric of $\mathbf{R}^N$.

Furthermore, the next claim asserts that the distribution kernel of Π_α is *non-negative* off the diagonal in $\Sigma_3 \times \Sigma_3$:

Claim 5.2. *On the boundary Σ_3, the operator Π_α is written in the form:*

$$\Pi_\alpha \varphi = P_\alpha \varphi + S_\alpha \varphi, \quad \varphi \in C^2(\Sigma_3),$$

where P_α is a first-order differential operator and

$$S_\alpha \varphi(x') = \int_{\Sigma_3} s(x', dy') \left[\varphi(y') - \sigma(x', y') \left(\varphi(x') + \sum_{i=1}^{N-1} \frac{\partial \varphi}{\partial x_i}(x')\,(y_i - x_i) \right) \right]$$

with a positive Borel kernel $s(x', dy')$.

Proof. By Theorem 2.2, it suffices to show the following:

$$x'_0 \in \Sigma_3, \varphi \in C^2(\Sigma_3), \varphi \geq 0 \text{ on } \Sigma_3 \text{ and } x'_0 \notin \operatorname{supp} \varphi \implies \Pi_\alpha \varphi(x'_0) \geq 0.$$

If we let

$$u = H_\alpha \varphi,$$

then we have

$$\begin{cases} (\alpha - A)u = 0 & \text{in } D, \\ u = \varphi & \text{on } \Sigma_2 \cup \Sigma_3. \end{cases}$$

Further, since $\varphi \geq 0$ on Σ_3, it follows from an application of the weak maximum principle (cf. [Ta2, Theorem A.1]) that

$$u \geq 0 \quad \text{in } D.$$

Hence this implies that

$$\Pi_\alpha \varphi(x'_0) = \frac{\partial u}{\partial \mathbf{n}}(x'_0) \geq 0,$$

since $u(x'_0) = \varphi(x'_0) = 0$. $\triangledown$

(2-b) The next claim asserts that, on the boundary Σ_2, the operator Π_α is a second-order, *degenerate elliptic differential operator*:

Claim 5.3. *On the boundary Σ_2, the operator Π_α is written in the form:*

$$\begin{aligned}\Pi_\alpha\varphi &= \frac{\partial}{\partial \mathbf{n}}(H_\alpha\varphi)\Big|_{\Sigma_2} \\ &= -\frac{1}{b}\left(\sum_{i,j=1}^{N-1} a^{ij}\frac{\partial^2\varphi}{\partial x_i \partial x_j} + \sum_{i=1}^{N-1} b^i \frac{\partial\varphi}{\partial x_i} + (c-\alpha)\varphi\right),\end{aligned}$$

where b is the Fichera function for the operator A. Note that

$$-\frac{1}{b}\left(\sum_{i,j=1}^{N-1} a^{ij}(x')\xi_i\xi_j\right) \geq 0 \quad \textit{on the cotangent bundle } T^*(\Sigma_2).$$

Proof. Let x_0' be an arbitrary point of Σ_2. We choose a local coordinate system $(x_1, \cdots, x_{N-1}, x_N)$ in a neighborhood of x_0' such that

$$\begin{cases} x_0' = 0, \\ D = \{x_N > 0\}, \\ \Sigma_2 = \{x_N = 0\}, \end{cases}$$

and assume that, in terms of this coordinate system, the operator A is written in the form

$$A = a^{NN}(x)\frac{\partial^2}{\partial x_N^2} + b^N(x)\frac{\partial}{\partial x_N} + \sum_{i,j=1}^{N-1} a^{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N-1} b^i(x)\frac{\partial}{\partial x_i} + c(x).$$

We remark that:

(i) $a^{NN}(0) = 0$ and $b^N(0) = b(0) < 0$, since $x_0' = 0 \in \Sigma_2$.
(ii) $a^{ij} \in C^\infty(\mathbf{R}^N)$, $a^{ij} = a^{ji}$ and

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geq 0, \quad x \in \mathbf{R}^N,\ \xi \in \mathbf{R}^N.$$

Since we have

$$\begin{cases} (A-\alpha)H_\alpha\varphi = 0 & \text{in } D, \\ H_\alpha\varphi = \varphi & \text{on } \Sigma_2 \cup \Sigma_3, \end{cases}$$

it follows that

$$\begin{aligned} 0 &= \alpha\varphi(0) - \left(a^{NN}(0)\frac{\partial^2}{\partial x_N^2}(H_\alpha\varphi)(0) + b^N(0)\frac{\partial}{\partial x_N}(H_\alpha\varphi)(0)\right. \\ &\quad \left. + \sum_{i,j=1}^{N-1} a^{ij}(0)\frac{\partial^2\varphi}{\partial x_i\partial x_j}(0) + \sum_{i=1}^{N-1} b^i(0)\frac{\partial\varphi}{\partial x_i}(0) + c(0)\varphi(0)\right) \\ &= \alpha\varphi(0) - \left(b^N(0)\frac{\partial}{\partial x_N}(H_\alpha\varphi)(0) + \sum_{i,j=1}^{N-1} a^{ij}(0)\frac{\partial^2\varphi}{\partial x_i\partial x_j}(0)\right. \end{aligned}$$

$$+ \sum_{i=1}^{N-1} b^i(0)\frac{\partial \varphi}{\partial x_i}(0) + c(0)\varphi(0)\Bigg).$$

This proves that

$$\begin{aligned}
&\Pi_\alpha \varphi(x_0') \\
&= \frac{\partial}{\partial x_N}(H_\alpha \varphi)(0) \\
&= -\frac{1}{b^N(0)}\left(\sum_{i,j=1}^{N-1} a^{ij}(0)\frac{\partial^2 \varphi}{\partial x_i \partial x_j}(0) + \sum_{i=1}^{N-1} b^i(0)\frac{\partial \varphi}{\partial x_i}(0) + (c(0)-\alpha)\varphi(0)\right) \\
&= -\frac{1}{b(x_0')}\left(\sum_{i,j=1}^{N-1} a^{ij}(x_0')\frac{\partial^2 \varphi}{\partial x_i \partial x_j}(x_0') + \sum_{i=1}^{N-1} b^i(x_0')\frac{\partial \varphi}{\partial x_i}(x_0') + (c(x_0')-\alpha)\varphi(x_0')\right). \triangledown
\end{aligned}$$

(3) By Claim 4.8 with $\psi = 1$, it follows that the function $\Pi_\alpha 1$ satisfies the condition:

$$\Pi_\alpha 1(x_0') = \frac{\partial}{\partial \mathbf{n}}(H_\alpha 1)(x_0') < 0, \quad x_0' \in \Sigma_2 \cup \Sigma_3.$$

Thus we find that

$$LH_\alpha 1(x_0') = \gamma(x_0') + \mu(x_0')\frac{\partial}{\partial \mathbf{n}}(H_\alpha 1)(x_0') - \alpha\delta(x_0') \leq 0, \quad x_0' \in \Sigma_2 \cup \Sigma_3.$$

Summing up, we have proved that the operator $LH_\alpha = Q_\alpha + \mu\, \Pi_\alpha$ is a second-order, classical pseudo-differential operator on the boundary $\Sigma_2 \cup \Sigma_3$, and satisfies all the conditions of Theorem 2.1.

(III) By applying Theorem 2.1, we obtain that the operator

$$\lambda I - LH_\alpha : C^{8+2\theta}\left(\Sigma_2 \cup \Sigma_3\right) \longmapsto C^{8+2\theta}\left(\Sigma_2 \cup \Sigma_3\right)$$

is surjective for $\lambda > 0$ sufficiently large. This verifies condition (5.1).

The proof of Theorem 5.1 is now complete. □

6. Proof of Main Theorem

The next theorem proves our Main Theorem:

Theorem 6.1. *Assume that the differential operator A satisfies hypotheses (H) and (E) and that the boundary condition L is* transversal *on $\Sigma_2 \cup \Sigma_3$. We define a linear operator*

$$\mathfrak{A} : C(\overline{D}) \longrightarrow C(\overline{D})$$

as follows.

(a) The domain $D(\mathfrak{A})$ of $\mathfrak{A}$ is the space

$$D(\mathfrak{A}) = \left\{u \in D(\overline{A});\ u|_{\Sigma_2 \cup \Sigma_3} \in \mathcal{D},\ Lu = 0 \text{ on } \Sigma_2 \cup \Sigma_3\right\}, \tag{6.1}$$

where $\mathcal{D}$ is the common domain of the operators $\overline{LH_\alpha}$.

(b) $\mathfrak{A}u = \overline{A}u,\ u \in D(\mathfrak{A})$.

Then the operator $\mathfrak{A}$ is the infinitesimal generator of some Feller semigroup on $\overline{D}$, and the Green operator $G_\alpha = (\alpha I - \mathfrak{A})^{-1}$ is given by the following formula:

$$G_\alpha f = G_\alpha^0 f - H_\alpha \left(\overline{LH_\alpha}^{-1} \left(\overline{LG_\alpha^0} f \right) \right), \quad f \in C(\overline{D}).$$

Proof. By virtue of Theorems 5.1 and 4.9, one can verify that the operator $\mathfrak{A}$, defined by formula (6.1), satisfies conditions (a) through (d) of Theorem 1.1, just as in the proof of [Ta2, Theorem 3.16]. Hence it follows from an application of part (ii) of the same theorem that the operator $\mathfrak{A}$ is the infinitesimal generator of some Feller semigroup on $\overline{D}$. $\square$

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